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LETTER TO THE EDITOR

Hidden quantum group structure in Chern–Simons theory

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Abstract. The unexpurgated K' matrix in the Chern–Simons theory of topological systems (such as the fractional Hall system, the chiral spin system and the anyon system) is viewed as a q-deformed Cartan matrix. The connection to the known generalized quantum groups is pointed out. An alternative interpretation in terms of quantum superalgebra in the graded Yang–Baxter basis also holds.

The $(2+1)$-dimensional Chern–Simons theory [1–8] has a number of interesting properties, for example, topological invariants [3], fractional statistics [4–7], link polynomials and knots [8], and connection to rational conformal field theory [8, 9]. Through the last two features, the connection with the Yang–Baxter equations and quantum groups [10, 11] is established.

Recently, Zee and his collaborators [12–14] have discussed the long-distance properties of two-dimensional topological fluids (such as the Hall fluid, the chiral spin fluid, and the anyon superfluid) in the Chern–Simons approach. The theory is characterized by a $m\times m$ K-matrix (see (4) below) which can be transformed into a K' matrix whose $(m-1)\times(m-1)$ block is the Cartan matrix for the Lie algebra $su(m)$. Thus a $SU(m)$ symmetry is claimed [12, 15] by ignoring the last row and the last column in the K' matrix.

In this letter we wish to point out that the unexpurgated K' matrix could be viewed as a q-deformed Cartan matrix which has been discussed in the generalized quantum groups [16]. This generalized quantum group structure arises in the non-standard braid group representations when the quantum group parameter q is changed into $-q^{-1}$ at certain strategic places in the Yang–Baxter R-matrix. In the conventional Yang–Baxter basis, the new algebra corresponds to a distorted $s\ell_q(m+1)$ with a special value of q (q being a root of unity). Alternatively, in the graded Yang–Baxter basis, the new algebra corresponds to the superalgebra $s\ell_q(m|1)$.

For the basic formalism of the K matrix in the Chern–Simons theory, we refer the reader to Zee [12]. The effective Lagrangian has the following form:

$$L=(1/4\pi)\varepsilon^{\mu\nu\lambda}\alpha_\mu K\partial_\nu\alpha_\lambda+\alpha^\mu j\mu \tag{1}$$

where α_μ is a gauge potential and j_μ is a reduced current (vortex current minus the electromagnetic current). K is the $m\times m$ matrix:

$$K=\begin{pmatrix} p+1 & p & \cdot & \cdot & p \\ p & p+1 & \cdot & \cdot & p \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ p & p & \cdot & \cdot & p+1 \end{pmatrix}. \tag{2}$$

Physically, the parameter p is a measure of the flux attached to each electron in the Hall effect; p enters in the fractional filling factor $\nu = m/(mp+1)$, for even p. In [12, 14], it is shown that the Fourier transform J_n of the J_0 current in the K-matrix Chern–Simons model satisfies the Kac–Moody algebra

$$[J^I_m, J^J_n] = m\delta_{m,-n}K^{IJ}. \tag{3}$$

Furthermore, the theory is invariant under a transformation on K, namely X^TKX with integer-valued matrix $X\,\varepsilon s\ell(m,Z)$ which would preserve the integer-valued topological vorticity. One finds [15, 14, 12] that

$$K' = X^TKX = \begin{pmatrix} 2 & -1 & 0 & \cdot & 0 \\ -1 & 2 & -1 & 0 & \cdot \\ 0 & -1 & 2 & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2 & -1 \\ 0 & \cdot & \cdot & -1 & p+1 \end{pmatrix} \tag{4}$$

by taking

$$X = \begin{pmatrix} 1 & 0 & \cdot & \cdot & \cdot \\ -1 & 1 & 0 & \cdot & \cdot \\ 0 & -1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & 1 \end{pmatrix}. \tag{5}$$

When the last row and the last column of the K' matrix are disregarded, one recognizes the $(m-1)\times(m-1)$ submatrix as the Cartan matrix for $su(m)$, thus a $SU(m)$ symmetry for the model [12, 14, 15].

Consider the unexpurgated $m\times m$ K' matrix given by (4). Equation (4) implies that the mth root vector (of the underlying algebra) has a norm $[(p+1)/2]^{1/2}$ instead of the usual 1. We can rescale this norm to be one, but at the cost of deforming its scalar product from $2\cos\theta = -1$ to $2\cos\theta = -[2/(p+1)]^{1/2}$. The rescaled K' matrix reads

$$K' = \begin{pmatrix} 2 & -1 & 0 & \cdot & \cdot & \cdot \\ -1 & 2 & -1 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 2 & -[2/(p+1)]^{1/2} \\ \cdot & \cdot & \cdot & \cdot & -[2/(p+1)]^{1/2} & 2 \end{pmatrix}. \tag{6}$$

A special class of the q-deformed Cartan matrix has been discussed in [16] in the context of non-standard braid group representations of the quantum group $s\ell_q(m)$. We here discuss the non-trivial case $p \neq 1$. (Physically relevant cases are when p is even.)

We go to the non-standard braid group representation [16] of $s\ell_q(m+1)$ by making one deformation $q \to -1/q$ in the last entry in the $(m+1)^2 \times (m+1)^2$ R-matrix. The net result is the following generalized algebra:

(*a*) $\qquad (X^{\pm}_m)^2 = 0 \qquad$ for the last mth element. $\qquad$ (7*a*)

(b) Corresponding to the regular Cartan matrix element $a_{ij}=3\delta_{ij}-1$, $|i-j|\leqslant 1$, for i, $j=1,\ldots,m-1$, ($a_{ij}=0$, $|i-j|>1$), we have the standard quantum algebra $s\ell_q(m)$:

$$K_iX_j^{\pm}K_i^{-1}=q^{\pm a_{ij}/2}X_j^{\pm} \qquad i,j=1,\ldots,m-1. \tag{7b}$$

(c) Corresponding to the entry $a_{m-1,m}$, we obtain

$$K_jX_i^{\pm}K_j^{-1}=(-q)^{\pm 1/2}X_i^{\pm} \tag{7c}$$

$$=q^{\pm 1/2\, a}m-1,X_i^{\pm} \qquad i,j=m-1,m. \tag{7d}$$

(d) Inserting the value from (6)

$$a_{m-1,m}=-[2/(p+1)]^{1/2} \tag{8}$$

we see that ($7c$) and ($7d$) are compatible for q being a root of unity:

$$q=\exp(-\mathrm{i}\pi/\{1+[2/(p+1)]^{1/2}). \tag{9}$$

This shows that the unexpurgated K' matrix of (4) can be interpreted as a q-deformed Cartan matrix which can be accommodated in the non-standard braid group representation $s\ell_q(m)$ with special value of q given by (9). Alternatively, in the graded Yang–Baxter basis, the non-standard braid group representations can be reinterpreted as quantum superalgebra [16, 17]. Thus for the present case of (6), we would get the quantum supersymmetry $SL_q(m|1)$. Such supersymmetry is perhaps not a great surprise for the anyon systems. A concrete realization of generalized quantum group structure in two-dimensional quantum fluids would be of interest and the details remain to be worked out.

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